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A high dimensional model representation based numerical method for solving ordinary differential equations

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Abstract A new numerical method for solving ordinary differential equations by using High Dimensional Model Representation (HDMR) has been developed in this work. Higher order ordinary differential equations can be reduced to a set of first order ODEs. Although HDMR is generally used for multivariate functions, univariate functions are taken into account throughout the work because of the ODEs' natures. Not the numerical solution but its image under an appropriately chosen linear ordinary differential operator is expressed as a linear combination of the positive deviation powers of independent variable from its initial value. The linear combination of these image functions are expected to form a basis set under consideration. The unknown constants in the linear combination are found by maximizing the constancy measurer formed in terms of the HDMR components after they are evaluated. Results are compared with well-known step size based numerical methods. A semi qualitative error analysis of the proposed method is also established.

Keywords High dimensional model representation · Interpolation · Orthogonal polynomials · Ordinary differential equations

Mathematics Subject Classification (2000) 65L05

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1 Introduction

This work focuses on the solution of ordinary differential equations accompanied by certain conditions to get unique solutions. For simplicity we study linear ODEs. However, it is possible to extend the case to nonlinear ODEs by accepting certain number of nonlinear algebraic equations to get the solution. Our main emphasis is to use orthonormal polynomial spline based Hilbert space which is a linear mathematical object, without mixing the effects of nonlinearities on the method. We want to show the efficiency of the method first on the linear ODEs. The nonlinear cases are left for a future study.

We solve initial value problems of ODEs. It is not a great limitation to deal with the first order since any higher order ODE can be converted to a first order set of ODEs after using appropriate redefinitions. The initial condition limitation can also be easily removed by changing the algebraic equations corresponding to initial conditions with their boundary equation counterparts.

The numerical solution of the considered problem is assumed as a linear combination of certain functions chosen in an appropriate Hilbert Space. These functions may be orthogonal polynomials or solely some integer powers of independent variable. However we need to use not directly these functions but their images under an appropriately chosen linear ordinary differential operator (ODO). The coefficients of the abovementioned linear combination are determined to maximize constancy of HDMR for the image of the ODE solution under the ODO told above. The ODO is constructed in such a way that the image of the true ODE solution under this ODO becomes just a constant, beyond that, 1 for getting high quality in HDMR constancy maximization. HDMR is a divide-and-conquer algorithm which can be used to approximate a given multivariate function by keeping the components involving less number of variables and ignoring the remaining ones. This method was first constructed by Sobol [1]. Later Rabitz and his colleagues extended the concept to the case of intervals other than unit closed interval between 0 and 1, and, the varying weights with the independent variable instead of the constant one and applied it to more general functions [2–12]. Demiralp and his group brought the Hilbert space and orthogonality concept together with the additivity measurer definitions to HDMR and constructed different type of new special purpose versions of HDMR, like Factorized HDMR, Logarithmic HDMR, Generalized HDMR, Transformational HDMR, Hybrid HDMR and so on [13–22].

The additivity measurers measure the norm square contribution of the terms retained at a specified multivariance level. Perhaps the most important one of these entities is the so-called “Constancy Measurer”, which measures how constant the original function is and in fact is the ratio of the square of the constant HDMR component to the norm square of the original function under the same geometry and weight of the HDMR. As the original function approaches to a constant value everywhere in the HDMR geometry, this entity tends to be 1. Hence if one desires to make a function with flexible parameters as close to a constant as possible then the constancy measurer should be maximized with respect to these flexibilities in the structure of the considered function.

Here, because of the univariances of ODEs, we use univariate HDMR which is at the lowest multivariance level and is not trivial. This means that the only univariate HDMR component which depends on the independent variable of ODE is desired to be suppressed as much as possible.

The paper is organized as follows. The next section briefly focuses on High Dimensional Model Representation. In the third section, the numerical method based on HDMR Constancy Maximization is described. The fourth section contains the application of the method on some problems. Convergence Issues and Error Estimation is explained in the fifth section. In the sixth section the error estimation is investigated on the fine points. The seventh section is about the spline possibilities and avoiding the curse of integration via universality. The case of higher order ordinary differential equations is explained in eighth section. The last section which contains the concluding remarks finalizes the paper.

2 High dimensional model representation

We consider a multivariate function $f(x_1, \ldots, x_N)$. The High Dimensional Model Representation for this function can be written as [1],

$$f(x_1, \ldots, x_N) = f_0 + \sum_{i=1}^{N} f_i(x_i) + \sum_{\substack{i_1, i_2 = 1, \\ i_1 < i_2}}^{N} f_{i_1, i_2}\left(x_{i_1}, x_{i_2}\right) + \cdots + f_{12\cdots N}(x_1, \ldots, x_N), \quad (1)$$

where f_0 denotes the constant term, $f_i(x_i)$ stands for the univariate terms, $f_{i_1 i_2}\left(x_{i_1}, x_{i_2}\right)$ represents the bivariate terms, and so on. There are 2^N terms at the right hand side of (1). Those terms are in fact the orthogonal decomposition components of the original function and they satisfy the following equalities [13]:

$$\int_{a_i}^{b_i} dx_i\, w_i(x_i)\, f_{x_{i_1}, \ldots, x_{i_k}}\left(x_{i_1}, \ldots, x_{i_k}\right) = 0, \quad x_i \in \left\{x_{i_1}, \ldots, x_{i_k}\right\}, 1 \le i, k \le N. \quad (2)$$

Here $w_i(x_i)$ is the ith factor of the HDMR's global weight function which is defined by

$$w(x_1, \ldots, x_N) \equiv \prod_{i=1}^{N} w_i(x_i). \quad (3)$$

Each weight function factor is normalized in related interval $[a_i, b_i]$, where $1 \le i \le N$ as follows [11,12]:

$$\int_{a_i}^{b_i} dx_i w_i(x_i) = 1, \quad 1 \le i \le N. \tag{4}$$

Using the Eqs. (2), (3) and (4), we can obtain the HDMR components of the multivariate function $f(x_1, \ldots, x_N)$ as follows:

$$f_0 = \int_{a_1}^{b_1} dx_1 \cdots \int_{a_N}^{b_N} dx_N w(x_1, \ldots, x_N) f(x_1, \ldots, x_N) \tag{5}$$

$$\begin{aligned} f_i(x_i) = & \int_{a_1}^{b_1} dx_1 w_1(x_1) \cdots \int_{a_{i-1}}^{b_{i-1}} dx_{i-1} w_{i-1}(x_{i-1}) \int_{a_{i+1}}^{b_{i+1}} dx_{i+1} w_{i+1}(x_{i+1}) \cdots \\ & \times \int_{a_N}^{b_N} dx_N w_N(x_N) f(x_1, \ldots, x_N) - f_0, \quad 1 \le i \le N \end{aligned} \tag{6}$$

$$\begin{aligned} f_{i_1 i_2}(x_{i_1}, x_{i_2}) = & \int_{a_1}^{b_1} dx_1 w_1(x_1) \cdots \int_{a_{i_1-1}}^{b_{i_1-1}} dx_{i_1-1} w_{i_1-1}(x_{i_1-1}) \\ & \times \int_{a_{i_1+1}}^{b_{i_1+1}} dx_{i_1+1} w_{i_1+1}(x_{i_1+1}) \cdots \\ & \times \int_{a_{i_2-1}}^{b_{i_2-1}} dx_{i_2-1} w_{i_2-1}(x_{i_2-1}) \int_{a_{i_2+1}}^{b_{i_2+1}} dx_{i_2+1} w_{i_2+1}(x_{i_2+1}) \cdots \\ & \times \int_{a_N}^{b_N} dx_N w_N(x_N) f(x_1, \ldots, x_N) \\ & - f_{i_1}(x_{i_1}) - f_{i_2}(x_{i_2}) - f_0, \quad 1 \le i_1 < i_2 \le N \end{aligned} \tag{7}$$

The remaining HDMR components can be obtained in the same way. We use the following inner product definition to specify the orthogonality as the square root of the inner product of a function by itself in the considered Hilbert space. The HDMR components are mutually orthogonal as it has proven in the last decade [13].

$$(u, v) \equiv \int_{a_1}^{b_1} dx_1 \ldots \int_{a_N}^{b_N} dx_N w(x_1, \ldots, x_N) u(x_1, \ldots, x_N) v(x_1, \ldots, x_N), \tag{8}$$

where u and v are any two functions in the Hilbert Space. Therefore we can write the norm square of a function u via (8) in the following way

$$\|u\|^2 = (u,u) = \int_{a_1}^{b_1} dx_1 \ldots \int_{a_N}^{b_N} dx_N w(x_1,\ldots,x_N)\, u(x_1,\ldots,x_N)^2 \tag{9}$$

By using this norm definition and the orthogonality conditions for HDMR components, we can write the following equality by taking the norm square of both sides of (1).

$$\|f\|^2 = \|f_0\|^2 + \sum_{i_1=1}^{N} \|f_i\|^2 + \sum_{\substack{i_1,i_2=1,\\ i_1<i_2}}^{N} \left\|f_{i_1 i_2}\right\|^2 + \cdots + \|f_{12\ldots N}\|^2, \tag{10}$$

which enables us to measure the quality of a truncated HDMR. The ratio of the norm square of the constant term to the norm square of the multivariate function is called the "Constancy Measurer"or "Zeroth Order Additivity Measurer"and denoted by σ_0. In general the k-th order additivity measurer is defined as follows:

$$\sigma_k = \frac{1}{\|f\|^2} \sum_{\substack{i_1,\ldots,i_k=1,\\ i_1<\cdots<i_k}}^{N} \left\|f_{i_1,\ldots,i_k}\right\|^2 + \sigma_{k-1} \tag{11}$$

All these measurers take values between 0 and 1 (inclusive) and they form a well-ordered sequence, that is,

$$0 \le \sigma_0 \le \sigma_1 \cdots \le \sigma_N = 1. \tag{12}$$

If the constancy measurer of a function is very close to 1, then this function behaves almost constant and it is the simplest function. If the HDMR of the function is truncated after the constant term, then in the limiting case the constancy measurer becomes exactly 1. Therefore maximizing the constancy measurer will help us to get better HDMR truncation quality at the constancy limit via considering function as a constant at least in a limiting case.

3 HDMR constancy maximization for the ODE solution's image under an appropriately chosen operator

There have been recent fruitful attempts to maximize the HDMR constancy for determination of various unknowns related to multivariate and even univariate functions. The constancy maximization can work well if we try to find a specific structure which is constant everywhere on an interval in a family of multivariate or univariate functions as long as the family has constant function members. This implies that an ordinary

differential equation should be converted to a relation where the solution of the ODE has a constant image under an appropriately defined operator which will be defined certainly with the aid of the ODE under consideration. Here we will consider only linear ODEs to get facilities from the linear vector spaces and algebra. The adding nonlinearity is a different issue and our purpose here to construct an efficient method based on HDMR. We are going to focus on a single general first order linear ODE as follows

$$y'(x) + p(x)y(x) = q(x), \quad y(a) = y_0 \tag{13}$$

whose exact solution can be of course found as long as the involved integration can be performed analytically [23]. Here we are not going to use this structure. Instead we will write

$$\mathcal{L}y(x) \equiv \frac{y'(x)}{q(x)} + \frac{p(x)}{q(x)}y(x) = 1, \quad y(a) = y_0 \tag{14}$$

which enforces $q(x)$ not to have a zero in the interval where we will use HDMR. Since the initial condition is given at $x = a$ we choose the left endpoint of the HDMR interval as a as long as the ODE's solution evolves forward (in ascending values of the independent variable starting from $x = a$). The other endpoint will be taken as the first nodal point where the ODE solution's value will be evaluated. We will denote that point by b for the moment.

$\mathcal{L}$ in (14) is a linear operator under which the solution of the ODE in (13) has the unit constant function image everywhere and therefore throughout the interval $[a, b]$. However, this operator is not appropriate for our further analysis since (14) contains two separate equations, ODE and the accompanying initial condition. It is better to work with the vanishing initial condition since it can be much easier to reflect this condition to basis set we will use later. Beyond that the homogeneous equations are more helpful to deal with appropriate subspaces in Hilbert spaces since homogeneity is mandatory in the conditions for defining subspaces. Therefore we will define

$$y(x) \equiv z(x) + y_0 \tag{15}$$

and write the following equations instead of (14)

$$\mathcal{L}z(x) \equiv \frac{z'(x)}{q(x) - y_0 p(x)} + \frac{p(x)}{q(x) - y_0 p(x)}z(x) = 1 \tag{16}$$

by keeping in mind that $z(a) = 0$. We can further write

$$g(x) \equiv \mathcal{L}z(x) = 1. \tag{17}$$

This is the basic idea to apply HDMR constancy maximization. We know that $g(x)$ can be unit constant function when $z(x)$ becomes the solution of (16). The solution $z(x)$ lies in a Hilbert space of functions which are square integrable over the interval $[a, b]$

as long as the functions $p(x)$ and $q(x)$ and the constant y_0 permit (as long as they provide analyticity in a complex plane region of the independent variable x, where the interval $[a, b]$ resides as an interior straight line segment). This Hilbert space which is based on function analyticity can be spanned by the power set composed of nonnegative integer powers of $(x - a)$. Since $z(a) = 0$ always holds, $z(x)$ should lie in the above Hilbert space's infinite dimensional subspace spanned by positive integer powers of $(x - a)$ (hence this infinite subspace does not contain constant functions).

The HDMR of $g(x)$ is very simple because of $g(x)$'s univariance and can be given by the following expression

$$g(x) = g_0 + g_1(x) \tag{18}$$

where g_0 is in fact the mean value of the function $g(x)$ over the interval $[a, b]$ while the integral of $g_1(x)$ over the same interval should vanish. That is,

$$g_0 = \int_a^b dx g(x), \quad \int_a^b dx g_1(x) = 0 \tag{19}$$

Since $z(x)$ lies in the subspace spanned by the positive integer powers of $(x - a)$ it can be expressed as an infinite linear combination of these powers and the truncations by retaining some number of first terms in ascending positive integer powers of $(x - a)$. That is,

$$z(x) \approx \sum_{i=1}^{m} c_i (x - a)^i, \quad m = 1, 2, \ldots \tag{20}$$

which implies

$$g^{(app)}(x) \equiv \sum_{i=1}^{m} c_i u_i(x), \quad m = 1, 2, \ldots \tag{21}$$

where

$$u_i(x) \equiv \mathcal{L}(x - a)^i, \quad i = 1, 2, \ldots \tag{22}$$

We call $u_i(x)$s the ODE image of $(x - a)^i$.

Now it is time to evaluate the constancy measurer of $g^{(app)}(x)$

$$\sigma_0 = \frac{\left\| g_0^{(app)} \right\|^2}{\left\| g^{(app)} \right\|^2} = \frac{\left(\sum_{k=1}^{m} \frac{c_k}{b-a} \int_a^b dx u_k(x) \right)^2}{\sum_{k,\ell=1}^{m} \frac{c_k c_\ell}{b-a} \int_a^b dx u_k(x) u_\ell(x)}, \tag{23}$$

where we have used the fact that HDMR uses weight functions whose integrals are 1, since $1/(b-a)$ factors are existing for this. If we define

$$\mathbf{u}(x) \equiv [\, u_1(x) \ \ldots \ u_m(x) \,]^T \,, \quad \mathbf{v} \equiv \frac{1}{(b-a)} \int_a^b dx \mathbf{u}(x) \tag{24}$$

$$\mathbf{G} \equiv \frac{1}{(b-a)} \int_a^b dx \mathbf{u}(x) \mathbf{u}(x)^T \tag{25}$$

where $\mathbf{G}$ is the Gram matrix of the image basis set $u_1(x), \ldots, u_m(x)$ and therefore positive definite as long as these m number of functions are linearly independent (linear dependence may happen to exist if the functions $p(x)$ and $q(x)$ together with the parameter y_0 provide special structures for this).

Now we can write

$$\sigma_0 = \frac{\mathbf{c}^T \mathbf{v} \mathbf{v}^T \mathbf{c}}{\mathbf{c}^T \mathbf{G} \mathbf{c}}. \tag{26}$$

where

$$\mathbf{c} \equiv [\, c_1 \ \ldots \ c_m \,]^T \tag{27}$$

The positive definiteness of $\mathbf{G}$ (as long as it exists, the cases where the null space of $\mathbf{G}$ is not empty can be particulary investigated and are not so interesting because of their somehow triviality) enables us to write the following Cholesky decomposition

$$\mathbf{G} = \mathbf{L} \mathbf{L}^T \tag{28}$$

where $\mathbf{L}$ is a lower triangular matrix which is unique as long as the positive branch of the square roots appearing in the diagonal element determinations is chosen. Now we can define

$$\overline{\mathbf{c}} = \mathbf{L}^T \mathbf{c}, \quad \overline{\mathbf{v}} = \mathbf{L}^{-1} \mathbf{v} \tag{29}$$

and therefore write

$$\sigma_0 = \frac{\overline{\mathbf{c}}^T \overline{\mathbf{v}} \, \overline{\mathbf{v}}^T \overline{\mathbf{c}}}{\overline{\mathbf{c}}^T \overline{\mathbf{c}}}. \tag{30}$$

The above expression is a Rayleigh quotient whose all eigenvalues vanish except the one which is positive and therefore the maximum value of σ_0, as given below.

$$\sigma_0 = \overline{\mathbf{v}}^T \overline{\mathbf{v}}. \tag{31}$$

The specific $\overline{\mathbf{c}}$ maximizing the constancy measurer σ_0 should be proportional to $\overline{\mathbf{v}}$. Hence

$$\overline{\mathbf{c}} = \alpha \overline{\mathbf{v}}. \tag{32}$$

which implies

$$g_0^{(app)} = \mathbf{c}^T \mathbf{v} = \overline{\mathbf{c}}^T \overline{\mathbf{v}} = \alpha \overline{\mathbf{v}}^T \overline{\mathbf{v}} = \alpha \sigma_0 \tag{33}$$

Apparently $g_0^{(app)}$ is a constant and we want it to be 1. This urges us to get

$$\alpha = \frac{1}{\sigma_0}, \quad \mathbf{c} = \frac{1}{\sigma_0} \left[\mathbf{L}^T \right]^{-1} \overline{\mathbf{v}} = \frac{1}{\sigma_0} \left[\mathbf{L}^T \right]^{-1} \mathbf{L}^{-1} \mathbf{v} = \frac{1}{\sigma_0} \mathbf{G}^{-1} \mathbf{v} \tag{34}$$

and

$$\sigma_0 = \mathbf{v}^T \mathbf{G}^{-1} \mathbf{v} \tag{35}$$

The components of the vector $\mathbf{c}$ determines the function $z^{(app)}(x)$ which is an m-th degree truncation approximation to $z(x)$ in the subspace spanned by positive integer powers of $(x - a)$. Thus we can write

$$z^{(app)}(x) = \sum_{i=1}^{m} c_i (x-a)^i = \frac{1}{\sigma_0} \sum_{i=1}^{m} \left(\mathbf{e}_i^T \mathbf{G}^{-1} \mathbf{v} \right) (x-a)^i \tag{36}$$

and specifically

$$z^{(app)}(b) = \frac{1}{\sigma_0} \sum_{i=1}^{m} \left(\mathbf{e}_i^T \mathbf{G}^{-1} \mathbf{v} \right) (b-a)^i \tag{37}$$

where $\mathbf{e}_i$ stands for the Cartesian unit vector whose only nonzero element, 1, resides at the ith position. The last formula evaluates the approximate value of the ODE's solution at $x = b$ (hence after one step proceeding with the step size $(b - a)$). The approximation to the original ODE's solution, $y^{(app)}(b)$ can be evaluated through

$$y^{(app)}(b) = z^{(app)}(b) + y_0 \tag{38}$$

4 Illustrative implementations

In this section we will write HDMR components for the numerical solution of certain first order linear differential equations and find their constancy measurers by using our method whose mainlines have been explained above. Then we are going to compare the analytical and numerical solutions of those differential equations. We used Mathematica 5.2 and MUPAD for numerical calculations.

Example 1 The equation for the first illustrative problem is given as follows.

$$y'(x) + \frac{4x}{x^2+1} y(x) = \frac{x}{x^2+1}, \qquad y(2) = 1. \tag{39}$$

The exact (analytical) solution is

$$y = \frac{76 + 2x^2 + x^4}{4\left(1 + x^2\right)^2}. \tag{40}$$

HDMR Constancy Maximization based approximate solution to the problem in (39) for the interval [2, 7/3] is given in Table 1.

The same problem's approximate solution values at $x = 8/3$ are given in Table 2.

In these tables the calculations are realized by using scripts in public software MuPAD, Computer Algebra System (it is no longer public now) at mostly symbolic level although certain cases urged us to use numerical computation facilities of this package. The numerical precision is held at the level of 30 decimal digits for assuring not to be facing with error accumulations. The constancy measurer of this example for the case where $b = 7/3$ is given in Table 3.

As can be easily noticed constancy measurer values have implications about the quality of the approximation. The deviations from the exact values are diminishing parallel to the decreases in σ_0. To be sure about how the accuracy decreases or is influenced by the interval growth we give the case where the interval length is just 1 for the same example in Table 4.

Table 1 The comparison of the approximate values with the values of the analytic solution in Example 1 for the interval length 1/3

m	$y^{(app)}(b)$	Error
1	0.696582184	$<(1/3)^4$
2	0.701448992	$<(1/3)^9$
3	0.701471359	$<(1/3)^{14}$
4	0.701471462	$<(1/3)^{19}$
Ex.	0.701471462	–

Table 2 The comparison of the approximate values with the values of the analytic solution in Example 1 for the interval length 2/3

m	$y^{(app)}(b)$	Error
1	0.511979652	$<(2/3)^7$
2	0.534639295	$<(2/3)^{19}$
3	0.534990995	$<(2/3)^{30}$
4	0.534997123	$<(2/3)^{40}$
Ex.	0.534997185	–

Table 3 The constancy measurer values in Example 1 for the interval length 1/3

m	$y^{(app)}(b)$
1	0.980846565310
2	0.999850174923
3	0.999999383276
4	0.999999998424

Table 4 The comparison of the approximate values with the values of the analytic solution in Example 1 for the interval length 1

m	$y^{(app)}(b)$	Error
1	0.388437195274	$<(1/4)^2$
2	0.436081650398	$<(1/4)^3$
3	0.437449088568	$<(1/4)^5$
4	0.437498912114	$<(1/4)^9$
Ex.	0.437500000000	–

Table 5 Calculation of errors at $x = 7/3$ of the methods in Example 1

Method	Error
Euler	0.101471
Modified Euler	0.01951
Runge Kutta 4	0.000191218
Cash	7.46329×10^{-7}
Dormand-Prince	2.28337×10^{-5}
Fehlberg	4.61659×10^{-5}

Table 4 shows that the decrease in error is not caused by the powers of the interval length. This and the previous ones imply that the decrease should be proportional to the length with a proportionality constant decreasing as the order of the method increases. The interval length corresponds to internodal distance or step size in the discretization methods. Hence this method seems to be superior to other methods like Euler or Runge–Kutta methods. The parameter 1/4, whose powers are used to imply the level of error decrease is roughly chosen here just to give an idea. True error estimate should of course depend on the structure of the ODE under consideration. The local errors of the proposed methods with the well known methods Euler, Modified Euler, Runge Kutta 4 and other numerical methods are presented in Table 5. The efficiency of the proposed method can be observed by comparing the error values from Tables 1 and 5.

Example 2 The second illustrative problem consists a trigonometric function

$$y'(x) + \frac{2}{x}y(x) = \frac{\cos x}{x^2}, \quad y(\pi) = 0. \tag{41}$$

The exact (analytical) solution is

$$y = \frac{\sin x}{x^2}. \tag{42}$$

HDMR Constancy Maximization based approximate solution to the problem in (41) for the interval $[\,\pi, 7\pi/6\,]$ is given in Table 6.

No need to explain, the efficiency is same as before for this case also. The same problem's approximate solution values at $x = 4\pi/3$ are given in Table 7.

The constancy measurer of this example for the case where $b = 7\pi/6$ is given in Table 8.

In Table 9 local errors obtained by well-known numerical methods are presented.

Hence the proposed method is superior to the well-known methods when we compare the error results for this example.

Example 3 The last illustrative problem is given through the following equations:

$$y'(x) + 2y(x) = xe^{-2x}, \quad y(1) = 0. \tag{43}$$

The exact (analytical) solution is

$$y = \frac{1}{2}e^{-2x}\left(x^2 - 1\right). \tag{44}$$

HDMR Constancy Maximization based approximate solution to the problem in (43) for the interval $[\,1, 3\,]$ is given in Table 10.

The same problem's approximate solution values at $x = 5/3$ are given in Table 11.

Table 6 The comparison of the approximate values with the values of the analytic solution in Example 2 for the interval length $\pi/6$

m	$y^{(app)}(b)$	$\lvert Error\rvert$
1	−0.036906949	$<(\pi/6)^{12}$
2	−0.037221281	$<(\pi/6)^{20}$
3	−0.037220036	$<(\pi/6)^{28}$
4	−0.037220026	$<(\pi/6)^{38}$
Ex.	−0.037220026	–

Table 7 The comparison of the approximate values with the values of the analytic solution in Example 2 for the interval length $\pi/3$

m	$y^{(app)}(b)$	$\lvert Error\rvert$
1	−0.0449949201	$<(\pi/3)^{8}$
2	−0.0492887666	$<(\pi/3)^{14}$
3	−0.0493583645	$<(\pi/3)^{21}$
4	−0.0493575378	$<(\pi/3)^{28}$
Ex.	−0.0493575294	–

Table 8 The constancy measurer values in Example 2 for the interval length $\pi/6$

m	$y^{(app)}(b)$
1	0.960664614378
2	0.999920022496
3	0.999999964291
4	0.999999999882

Table 9 Calculation of errors at $x = \pi/6$ of the methods in Example 2

Method	Error
Euler	0.0158316
Modified Euler	0.00171833
Runge Kutta 4	3.61248×10^{-6}
Cash	1.50187×10^{-8}
Dormand-Prince	3.8137×10^{-7}
Fehlberg	6.54517×10^{-7}

Table 10 The comparison of the approximate values with the values of the analytic solution in Example 3 for the interval length 1/3

m	$y^{(app)}(b)$	Error
1	0.02758100354	$<(1/3)^6$
2	0.02703066604	$<(1/3)^{10}$
3	0.02702137040	$<(1/3)^{18}$
4	0.02702134215	$<(1/3)^{24}$
Ex.	0270213421422	–

Table 11 The comparison of the approximate values with the values of the analytic solution in Example 3 for the interval length 2/3

m	$y^{(app)}(b)$	Error
1	0.03377574668	$<(2/3)^6$
2	0.03184851488	$<(2/3)^8$
3	0.03171234044	$<(2/3)^{12}$
4	0.03171022315	$<(2/3)^{23}$
Ex.	0.03171021631	–

The constancy measurer of this example for the case where $b = 4/3$ is given in Table 12.

The local errors obtained by well-known numerical methods are shown in Table 13. The efficiency of the proposed method can be seen from these tables easily.

Table 12 The constancy measurer values in Example 3 for the interval length 1/3

m	$y^{(app)}(b)$
1	0.940240300915
2	0.999459548957
3	0.999998913832
4	0.999999999753

Table 13 Calculation of errors at $x = 1/3$ of the methods in Example 3

Method	Error
Euler	0.0180904
Modified Euler	0.00434727
Runge Kutta 4	0.0000970376
Cash	1.85546×10^{-7}
Dormand-Prince	1.02066×10^{-5}
Fehlberg	2.82015×10^{-5}

5 Convergence issues and error estimation

Reconsider the vector $\overline{\mathbf{v}}$ defined through (29) and (24). We can write

$$\overline{\mathbf{v}} = \mathbf{v} \equiv \frac{1}{(b-a)} \int_a^b dx \mathbf{L}^{-1} \mathbf{u}(x) \tag{45}$$

and define

$$\mathbf{u}_{ort}(x) \equiv \mathbf{L}^{-1} \mathbf{u}(x) \tag{46}$$

where the subscript *ort* implies orthogonality since these new vector components are mutually orthonormal. We can show this fact as follows

$$\begin{aligned} \frac{1}{(b-a)} \int_a^b dx \mathbf{u}_{ort}(x) \mathbf{u}_{ort}(x)^T &= \frac{1}{(b-a)} \int_a^b dx \mathbf{L}^{-1} \mathbf{u}(x) \mathbf{u}(x)^T \left[\mathbf{L}^T \right]^{-1} \\ &= \mathbf{L}^{-1} \mathbf{G} \left[\mathbf{L}^T \right]^{-1} = \mathbf{I} \end{aligned} \tag{47}$$

We can express the vector $\mathbf{u}_{ort}(x)$ explicitly as follows

$$\mathbf{u}_{ort}(x) \equiv \left[u_1^{(ort)}(x) \ \ldots \ u_m^{(ort)}(x) \right], \tag{48}$$

where

$$\frac{1}{(b-a)}\int_a^b dx u_i^{(ort)}(x)u_j^{(ort)}(x) = \delta_{i,j}, \quad i,j = 1,2,\ldots,m \tag{49}$$

and δ stands for the Kronecker delta symbol. Each $u_i^{(ort)}(x)$ function spans a one dimensional subspace of the entire Hilbert space. We can define the following projection operators to facilitate the further analysis

$$\widehat{P}_i f(x) \equiv \frac{u_j^{(ort)}(x)}{(b-a)}\int_a^b d\xi u_i^{(ort)}(\xi)\, f(\xi), \quad i = 1,2,\ldots \tag{50}$$

where $f(x)$ stands for an arbitrary function in the entire Hilbert space. If we denote the unit constant function by $u_c(x)$ then we can write

$$\begin{aligned}\sigma_0 &= \overline{\mathbf{v}}^T\overline{\mathbf{v}} = \frac{1}{(b-a)}\int_a^b dx\int_a^b d\xi \mathbf{u}_{ort}(x)^T\mathbf{u}_{ort}(\xi)\\ &= \frac{1}{(b-a)}\int_a^b dx u_c(x)\left(\sum_{i=1}^m \widehat{P}_i\right)u_c(x) = \frac{1}{(b-a)}\int_a^b dx u_c(x)\widehat{P}^{(m)}u_c(x)\\ &= 1 - \frac{1}{(b-a)}\int_a^b dx u_c(x)\left[\widehat{I}\widehat{P}^{(m)}\right]u_c(x)\end{aligned} \tag{51}$$

where $\widehat{P}^{(m)}$ projects from the entire Hilbert space to its m dimensional subspace spanned by $u_1^{(ort)}(x),\ldots,u_m^{(ort)}(x)$. If the infinite set of functions composed of $u_i^{(ort)}(x)(i = 1,\ldots)$ forms a complete basis set for the entire set then one can show that the operator $\left[\widehat{I} - \widehat{P}^{(m)}\right]$ (we call fluctuation operator [24,25]) approaches to zero operator when m goes infinity, because of Parseval inequality. If this completeness happens to exist, then the fluctuation operator above becomes a nonnegative operator which projects to the complement of the entire Hilbert space to the m dimensional subspace spanned by $u_1^{(ort)}(x),\ldots,u_m^{(ort)}(x)$. This however implies that σ_0, the constancy measurer, stays less than 1 and approaches to 1 as m grows unboundedly. The completeness issue mentioned above is a delicate subject and may require rigor in the mathematical analysis. However, the analyticity existing in the ODE and its solution in the interested interval will permit us to prove the completeness because of the power series expansion possibilities. We are not going to get into details and we are going to emphasize on the cases where the existence of completeness is assumed.

Now we consider the differential operator appearing in (16) and (22). As long as the ODE accompanied by a compatible initial condition in our consideration has a unique solution, $\mathcal{L}$ has a unique inverse. This enables us to revert (22) as follows

$$(x-a)^i = \mathcal{L}^{-1}u_i(x), \quad i = 1, 2, \ldots \tag{52}$$

which implies

$$\begin{aligned} z(x) &\approx \sum_{i=1}^{m} c_i \mathcal{L}^{-1}u_i(x) = \mathbf{c}^T \mathcal{L}^{-1}\mathbf{u}(x) = \frac{1}{\sigma_0}\left[\mathbf{L}^{-1}\mathbf{v}\right]^T \mathcal{L}^{-1}\left[L^{-1}\mathbf{u}(x)\right] \\ &= \frac{1}{\sigma_0}\left[\mathcal{L}^{-1}\mathbf{u}_{ort}(x)\right]\frac{1}{(b-a)}\int_a^b d\xi\, \mathbf{u}_{ort}(\xi)\, u_c(\xi) \\ &= \frac{1}{\sigma_0}\mathcal{L}^{-1}\widehat{P}^{(m)}u_c(x) \end{aligned} \tag{53}$$

As long as completeness exists in the spanning functions $\widehat{P}^{(m)}$ goes to unit operator over the entire Hilbert space and therefore $z(x)$ approaches to the solution of its differential equation. This completes the proof of the method's convergence.

6 On the fine points of the error estimation for the presented method

Reconsider the $\mathcal{L}$ operator given by (16). Its use in (22) produces

$$u_i(x) = \frac{i(x-a)^{i-1} + p(x)(x-a)^i}{q(x) - y_0 p(x)}, \quad i = 1, 2, \ldots \tag{54}$$

Our purpose now to investigate how the method's convergence changes with respect to the length of the HDMR interval. To this end it is better to make the following definitions

$$h \equiv b - a, \quad \xi \equiv \frac{x-a}{b-a} \longrightarrow x = a + h\xi \tag{55}$$

which enable us to rewrite (54) more explicitly in terms of h and ξ as follows

$$u_i(x) = h^{i-1}\frac{i\xi^{i-1} + hp(a+h\xi)\xi^i}{q(a+h\xi) - y_0 p(a+h\xi)}, \quad i = 1, 2, \ldots \tag{56}$$

Now we can write the following series expansion as long as the relevant functions permit (their analyticity assumption does this)

$$p(a+h\xi) = \sum_{j=0}^{\infty} p_j h^j \xi^j, \quad q(a+h\xi) = \sum_{j=0}^{\infty} q_j h^j \xi^j \tag{57}$$

where p_j and q_j denote the Taylor series coefficients of the related functions for the expansion point a. These expansions can be plugged in (56) to get the following formula

$$u_i(x,h) = \sum_{j=0}^{\infty} h^{j+i-1} u_i^{(j)}(\mathbf{p},\mathbf{q})\,\xi^{j+i-1}, \quad i = 1,2,\ldots,m \tag{58}$$

where we have emphasized on the h dependence of u_i functions by explicitly denoting it and the coefficients $u_i^{(j)}(\mathbf{p},\mathbf{q})$ which vary with the p and q coefficients having indices not greater than j bring the ODE dependence into the structure and this formula remains valid as long as the related functions of the expansion converge throughout the new HDMR interval $[0,1]$ not for x but now for ξ.

We can rewrite (58) in the following vector form to facilitate the further analysis

$$\mathbf{u}(a+h\xi) = \sum_{j=0}^{\infty} h^j \xi^j \mathbf{H}\mathbf{U}^{(j)}(\mathbf{p},\mathbf{q})\,\boldsymbol{\xi}, \tag{59}$$

where $\mathbf{H}$ stands for the $m \times m$ diagonal matrix whose diagonal elements are composed of one by one downward ascending nonnegative integer powers of h starting from 0 and $\mathbf{U}^{(j)}(\mathbf{p},\mathbf{q})$ is another diagonal matrix whose ith diagonal element is $u_i^{(j)}(\mathbf{p},\mathbf{q})$ for $i = 1,2,3,\ldots$ On the other hand, the ith element of $\boldsymbol{\xi}$, the new vector, is ξ^{i-1} for $i = 1,2,\ldots,m$.

The utilization of (59) in the second equality of (24) produces

$$\mathbf{v}(h) = \sum_{j=0}^{\infty} h^j \mathbf{H}\mathbf{U}^{(j)}(\mathbf{p},\mathbf{q})\,\boldsymbol{\xi}_j^{(int)}, \tag{60}$$

where $\boldsymbol{\xi}_j^{(int)}$ stands for the integral of the vector $\xi^j\boldsymbol{\xi}$ over the integral $[0,1]$, that is, its ith element is $1/(i+j)$ for $i = 1,2,\ldots$

The utilization of the above series expansion for $\mathbf{u}(x)$ in (25) allows us to get

$$\mathbf{G}(h) = \sum_{j=0}^{\infty} h^j \mathbf{H}\mathbf{G}_j(\mathbf{p},\mathbf{q})\,\mathbf{H}, \tag{61}$$

where

$$\mathbf{G}_j(\mathbf{p},\mathbf{q}) \equiv \sum_{k=0}^{j} \mathbf{U}^{(k)}(\mathbf{p},\mathbf{q})\,\boldsymbol{\Xi}_j\mathbf{U}^{(j-k)}(\mathbf{p},\mathbf{q}), \quad j = 0,1,2,\ldots \tag{62}$$

and

$$\boldsymbol{\Xi}_k \equiv \int_0^1 d\xi\,\xi^k \boldsymbol{\xi}\boldsymbol{\xi}^T, \quad [\boldsymbol{\Xi}_k]_{i,j} = \frac{1}{i+j+k-1}, \quad i,j = 1,2,\ldots,m, \quad k = 0,1,2,\ldots \tag{63}$$

A careful glance at the last three formulae reveals that

$$\mathbf{G}(h)^{-1} = \sum_{j=0}^{\infty} h^j \mathbf{H}^{-1} \mathbf{G}_j^{(-1)}(\mathbf{p}, \mathbf{q})\, \mathbf{H}^{-1} \tag{64}$$

$$\sum_{k=0}^{j} \mathbf{G}_k(\mathbf{p}, \mathbf{q})\, \mathbf{G}_{k-j}^{(-1)}(\mathbf{p}, \mathbf{q}) = \delta_{k,0}\mathbf{I}, \quad k = 0, 1, 2, \ldots \tag{65}$$

where the $\delta_{k,0}$ denotes again Kronecker delta symbol while $\mathbf{I}$ stands for $m \times m$ unit matrix.

The combination of (60) and (64) produces

$$\begin{aligned} \mathbf{G}(h)^{-1}\mathbf{v}(h) &= \sigma_0(h)\mathbf{c}(h) \\ &= \sum_{j=0}^{\infty} h^j \sum_{k=0}^{j} \mathbf{H}^{-1}\mathbf{G}_k^{(-1)}(\mathbf{p}, \mathbf{q})\, \mathbf{U}^{(j-k)}(\mathbf{p}, \mathbf{q})\, \boldsymbol{\xi}_{j-k}^{(int)}, \end{aligned} \tag{66}$$

which urge us to write

$$\sigma_0(h) \equiv \sum_{i=0}^{\infty} \sigma_i^{(0)} h^i, \quad \mathbf{c}(h) \equiv \sum_{i=0}^{\infty} h^i \mathbf{c}_i \tag{67}$$

where

$$\begin{aligned} \sigma_i^{(0)} \equiv \sum_{j=0}^{i} \boldsymbol{\xi}_j^{(int)^T} \mathbf{U}^{(j)}(\mathbf{p}, \mathbf{q}) \sum_{k=0}^{i-j} \mathbf{G}_k^{(-1)}(\mathbf{p}, \mathbf{q})\, \mathbf{U}^{(i-j-k)}(\mathbf{p}, \mathbf{q})\, \boldsymbol{\xi}_{i-j-k}^{(int)}, \\ i = 0, 1, 2, \ldots \end{aligned} \tag{68}$$

and

$$\begin{aligned} \sum_{j=0}^{i} \sigma_j^{(0)} \mathbf{c}_{i-j} = \sum_{k=0}^{i} \mathbf{H}^{-1}\mathbf{G}_k^{(-1)}(\mathbf{p}, \mathbf{q})\, \mathbf{U}^{(i-k)}(\mathbf{p}, \mathbf{q})\, \boldsymbol{\xi}_{i-k}^{(int)}, \\ i = 0, 1, 2, \ldots \end{aligned} \tag{69}$$

Now a simple but careful analysis shows that $\sigma_0^{(1)}$ takes the value of 1 without depending on the ODE under consideration. This is because all ODE dependent entities cancel out at $h = 0$ limit. This may be considered somehow trivial but the good news is the possibility of achieving exact solution at this limit (The exact solution at this limit is just the given initial value and it is obtained from this since $\mathbf{c}$ vanishes). The ODE dependence enter the structure as we proceed to the higher order expansion coefficients. To see this we can focus on $\sigma_1^{(0)}$ and write the following equalities

$$\mathbf{G}_0(\mathbf{p},\mathbf{q}) = \mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})\,\boldsymbol{\Xi}_0\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q}) \tag{70}$$

$$\mathbf{G}_1(\mathbf{p},\mathbf{q}) = \mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})\;\boldsymbol{\Xi}_1\mathbf{U}^{(1)}(\mathbf{p},\mathbf{q}) + \mathbf{U}^{(1)}(\mathbf{p},\mathbf{q})\,\boldsymbol{\Xi}_1\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q}) \tag{71}$$

$$\mathbf{G}_0^{(-1)}(\mathbf{p},\mathbf{q}) = \mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1}\,\boldsymbol{\Xi}_0^{-1}\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1} \tag{72}$$

$$\begin{aligned}\mathbf{G}_1^{(-1)}(\mathbf{p},\mathbf{q}) = &-\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1}\,\boldsymbol{\Xi}_0^{-1}\boldsymbol{\Xi}_1\mathbf{U}^{(1)}(\mathbf{p},\mathbf{q})\,\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1}\\ &\times\boldsymbol{\Xi}_0^{-1}\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1} - \mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1}\,\boldsymbol{\Xi}_0^{-1}\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1}\\ &\times\mathbf{U}^{(1)}(\mathbf{p},\mathbf{q})^{-1}\,\boldsymbol{\Xi}_1\boldsymbol{\Xi}_0^{-1}\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1}\end{aligned} \tag{73}$$

By using these formulae we can obtain the following result by skipping some intermediate details

$$\begin{aligned}&\boldsymbol{\xi}_0^{(int)^T}\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})\,\mathbf{G}_0^{(-1)}(\mathbf{p},\mathbf{q})\,\mathbf{U}^{(1)}(\mathbf{p},\mathbf{q})\,\boldsymbol{\xi}_1^{(int)}\\ &= \mathbf{e}_1^T\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1}\,\mathbf{U}^{(1)}(\mathbf{p},\mathbf{q})\,\boldsymbol{\xi}_1^{(int)}\\ &= -\frac{y_0p(a)^2 - q(a)p(a) + q'(a) - y_0p'(a)}{2q(a) - 2y_0p(a)}\end{aligned} \tag{74}$$

where we have used the fact that $\boldsymbol{\Xi}_0^{(-1)}\boldsymbol{\xi}_0^{(int)}$ is an m-element vector which is identical to the first cartesian unit vector $\mathbf{e}_1$ whose only nonzero element is 1 residing at its first elemental position. We can also similarly write the following equality

$$\begin{aligned}&\boldsymbol{\xi}_0^{(int)^T}\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})\,\mathbf{G}_1^{(-1)}(\mathbf{p},\mathbf{q})\,\mathbf{U}^{(1)}(\mathbf{p},\mathbf{q})\,\boldsymbol{\xi}_0^{(int)}\\ &= -2\mathbf{e}_1^T\,\boldsymbol{\Xi}_1\mathbf{U}^{(1)}(\mathbf{p},\mathbf{q})\,\mathbf{U}^{(0)}(\mathbf{p},\mathbf{q})^{-1}\,\mathbf{e}_1\\ &= \frac{y_0p(a)^2 - q(a)p(a) + q'(a) - y_0p'(a)}{q(a) - y_0p(a)}\end{aligned} \tag{75}$$

Doubling the both sides of (74) and side by side adding the resulting equality to (75) produces $\sigma_1^{(0)} = 0$ at the left hand side of the concluded equality while the final right hand side vanishes. That is

$$\sigma_1^{(0)} = 0 \tag{76}$$

which means that the first correction to the $\sigma_0^{(app)}$ parameter's h dependence comes not from h but h^2 containing terms in order analysis with respect to h.

Similar analysis can be driven for the evaluation of other and higher order terms. Since our purpose here has not been to report the rather technical details and the conceptuality was the most important thing we suffice these formulae. However the important thing is the fact that for the solution and the constancy measurer and the other entities it is possible to construct explicit analytical structures maybe not via handy calculations but symbolic programming. This stage is universal since we do not use ODE dependent agents numerically (instead we do symbolically). Once it is done, the results can be stored in a database and then used indefinitely many times for different ODEs from the same family on which formulae hold. This means that what

it is done in Runge–Kutta and similar methods depending on order can be realized. Of course the resulting formulae will differ in ours and other cases. However this is quite natural and is very possibly the reason why some superiority of the present method is observed versus the others as the numerical experimentations show.

The first and higher order derivative values can be avoided with the aid of the values of the corresponding entities at $x = b$ or some other interval points. The higher order expansion terms the more derivatives or more values at different points of the HDMR interval. As all our numerical experimentations show the method gets certain approximate σ_0 values obtained by truncating the h-power expansions such that the values start from a value less than 1 for the first order retainment and that value gets closes as the more higher powers of h is involved. Hence, it is possible to develop a stepwise approach very similar to the higher order Runge–Kutta Methods [26–30] as we mentioned above.

7 Spline possibilities and avoiding the curse of integration via universality

The method developed here has been based on HDMR constancy maximization and its convergence, under certain conditions, has been shown above. Here remains to talk about its computational cost. First of all it is based on matrix algebraic operations which may cause cubic increase in the computational cost. Although this may not be considered a harsh limitation since the lower dimensions like four (even two) seem to be quite powerful. The volume of the matrix algebraic operations is completely determined by the interval length and the behaviors of of the ODE agents like coefficient functions or conditioning parameters. On the other hand all integrals appearing in the vector and matrix elements approach some constant values depending on the ODE agents' values at the lowest limit of the integration when the interval length diminishes to zero. These approximations of course weakens the method in the numerical quality. However it is possible to construct various nodes inside the interval such that two consecutive nodes, including the end point of the HDMR interval, make disjoint subintervals whose union gives the entire HDMR interval. As sufficiently high number of nodes are constructed, the lengths of individual subintervals can be made as less as desired. This facilitates the construction of a some how discretized method like Runge–Kutta methods and the others. In this approach the integrals can be constructed by taking not only the dominant terms but the descending contributions. However, here arises the question "how many terms in the integrations and how small subinterval lengths should be used in the method". This can be based on an order analysis like the other discretized method do. Then some order of approach can be mentioned in application like mth order Runge–Kutta method. These may somehow use spline structures alt hough we are not willing to emphasize on them here. We find it enough to mention about this subject at this level since we prefer to focus on global case where just a single interval is considered and inside it the unknown function of the ODE under consideration is evaluated in terms of certain basis functions like positive powers of independent variable deviations from the left end of the interval.

In the global case the curse of the integration appears on the method. Integration is of course one of the most expensive operation in the sense of numerical calculations.

Nevertheless, it is possible to introduce universality to it like Gauss quadrature [31] and fluctuation free integration [32] do. The nodes and the weight constants of these structures are universal, in other words, they do not depend on the structure of the function to be integrated. They depend on the weight function and limit points of the integration. The nodes are evaluated as the eigenvalues of independent variable's truncated matrix representation, or the so-called Jacobi matrix of the Gauss quadrature. The weight constants are evaluated as the squares of the corresponding normalized eigenvectors of those matrices. Once they are evaluated they can be used indefinitely many times for integrals with different functions to be integrated but same intervals and weight functions. We do not attempt to give explicit formulae here since it just straightforward and a rather technical issue.

8 Numerical solution of higher order ODEs

In this section we are going to find a numerical solution for an nth order linear ODE, which has the general form

$$\begin{aligned} &y^{(n)} + a_1(x)\, y^{(n-1)} + \cdots + a_n(x)\, y = q(x) \\ &y(x_1) = y_0, \quad y'(x_1) = y_1, \ldots, y^{(n-1)}(x_1) = y_{n-1}. \end{aligned} \tag{77}$$

This problem can be transformed to a set of first order ODE by defining suitable functions as follows.

$$\begin{aligned} y' &= z_1 \\ z_1' &= z_2 \\ z_2' &= z_3 \\ &\vdots \\ z_{n-2}' &= z_{n-1} \\ y_{n-1}' &= -a_1(x)\, z_{n-1} - \cdots - a_n(x)\, y + q(x) \end{aligned} \tag{78}$$

The initial conditions then can be written as

$$y(x_1) = y_0, z_1(x_1) = y_1, \ldots, z_{n-1}(x_1) = y_{n-1}. \tag{79}$$

Therefore the problem in (77) can be expressed in a vector form.

$$\mathbf{p}' = \mathbf{q}, \quad \mathbf{p}(x_1) = \mathbf{q}_0 \tag{80}$$

The rest of the ODE solution can be conducted by using our method developed here, by taking care of dealing with not scalars but vectors and matrices and therefore possibly noncommutative items. We are not going to attempt to give more details and illustrative examples since there is almost nothing enthusiastic except the routine dimension increase.

9 Concluding remarks

We developed a new method for numerically approximating the solutions of linear differential equations in this study. We have taken first order linear ODEs at the focus for simplicity. We observed that the method shows satisfactorily good approximations to the exact solutions when we applied the method on certain problems, even by using small number of basis functions. We itemize certain concluding remarks below.

- The method is based on HDMR constancy maximization. However, HDMR is used not for the considered ODE's solution but its image under an appropriate ODO (ordinary differential operator) which is constructed by completely reflecting the structure of ODE under consideration such that the image becomes a constant function and preferably unit constant function by giving the chance of enforcing constancy dominance in HDMR. This philosophy is not peculiar to ODEs only any linear equation(s) can be converted as the constancy maximization of the HDMR for an image of the solution of that equation under an appropriately defined linear operator;
- Focusing on just scalar first order linear ODEs which can be somehow considered trivial does not mean any great loss of generality because we can deal with the first order vector or matrix ODEs by replacing certain scalar entities with vectors or matrices depending on the roles of those entities in the ODEs under consideration. Thus it is quite easy to extend this method to the linear vector or matrix ODEs;
- In this paper we have dealt with the ODEs with initial conditions. The method can be equivalently applied on the ODEs with the boundary conditions and even mixed ones in the vector or matrix ODE cases. The only thing to do is to replace the algebraic conditions coming from the initial conditions by the ones coming from the boundary conditions;
- ODEs with boundary conditions appear generally in two different forms: (1) There is no unknown proportionality parameter in the structure, then, the problem's solution necessitates the inversion of the ODEs differential operator. That is, it is an inversion problem; (2) in the second case there is an unknown proportionality constant, then, the problem's solution produces, generally, infinite number of eigenmodes (eigenvalues and eigenfunctions). This case is not considered here although we intensely focused on the issue in our group studies;

The above items imply that this article reports about a rather preliminary but strongly original conceptual stage of a wide area research. However all implementations are quite encouraging us to extend the research to the other cases which are mentioned but not attempted to apply our method on. The fine points of an error analysis also shows a strong and widely applicable theory from just now.

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References

1. I.M. Sobol, Sensitivity estimates for nonlinear mathematical models. Math. Model. Comput. Exp. **1**, 407–414 (1993)

2. H. Rabitz, O. Alis, General foundations of high dimensional model representations. J. Math. Chem. **25**, 197–233 (1999)
3. H. Rabitz, O. Alis, J. Shorter, K. Shim, Efficient input-output model representations. Comput. Phys. Comm. **117**, 11–20 (1999)
4. H. Rabitz, O. Alis, in *Managing the tyranny of parameters in mathematical modelling of physical systems*, ed. by K. Chan, M. Scott Sensitivity Analysis (John Wiley and Sons, Chichester, 2000)
5. G. Li, S.W. Wang, C. Rosenthal, H. Rabitz, High dimensional model representations generated from low dimensional data samples. I. mp-Cut-HDMR. J. Math. Chem. **30**, 1–30 (2001)
6. G. Li, M. Artamonov, H. Rabitz, S.W. Wang, P.G. Georgopoulos, M. Demiralp, High dimensional model representations generated from low dimensional terms-LP-RS-HDMR. J. Comput. Chem. **25**, 1149–1156 (2002)
7. G. Li, S.W. Wang, H. Rabitz, Practical approaches to construct rs-hdmr component functions. J. Phys. Chem. A. **106**, 8721–8733 (2002)
8. G. Li, J. Schoendorf, T. Ho, H. Rabitz, Multicut-HDMR with an application to an ionospheric model. J. Comput. Chem. **25**, 1149–1156 (2004)
9. M.Y. Hayes, B. Li, H. Rabitz, Estimation of molecular properties by high dimensional model representation. J. Phys. Chem. A. **110**, 264–272 (2006)
10. G. Li, S.W. Wang, P.G. Georgopoulos, J. Schoendorf, H. Rabitz, Random sampling high dimensional model representation (RS-HDMR) and orthogonality of its different order component functions. J. Phys. Chem. A. **110**, 2474–2485 (2006)
11. H. Rabitz, O. Alis, Efficient implementation of high dimensional model representations. J. Math. Chem. **29**, 127–142 (2001)
12. G. Li, C. Rosenthal, H. Rabitz, High dimensional model representations. J. Phys. Chem. A **105**, 7765–7777 (2001)
13. M. Demiralp, High dimensional model representation and its application varieties. in *Proceedings of the Fourth International Conference on Tools for Mathematical Modelling*, vol. 9 (2003), pp. 146–159
14. M.A. Tunga, M. Demiralp, A factorized high dimensional model representation on the nodes of a finite hyperprismatic regular grid. Appl. Math. Comput. **164**, 865–883 (2005)
15. M.A. Tunga, M. Demiralp, A factorized high dimensional model representation on the partitioned random discrete data. Appl. Numer. Anal. Comput. Math. **1**, 231–241 (2004)
16. B. Tunga, M. Demiralp, Hybrid high dimensional model representation approximants and their utilization in applications. Math. Res. **9**, 438–446 (2003)
17. M.A. Tunga, M. Demiralp, Hybrid high dimensional model representation (HHDMR) on the partitioned data. J. Comput. Appl. Math. **185**, 107–132 (2006)
18. M.A. Tunga, M. Demiralp, Data partitioning via generalized high dimensional model representation (GHDMR) and multivariate interpolative applications. in *Proceedings of the Fourth International Conference on Tools For Mathematical Modelling*, vol 9 (2003), pp. 447–462
19. N. Sen, N.A. Baykara, M. Demiralp, High dimensional model representation (HDMR) and trigonometric transformational HDMR. in *Proceedings of the 1st WSEAS International Conference on Multivariate Analysis and its Application in Science and Engineering (MAASE'08)*, (2008), pp. 209–215
20. B. Tunga, M. Demiralp, A novel hybrid high dimensional model representation (HHDMR) based on the combination of plain and logarithmic high dimensional model representations. in *Proceedings of the 12th WSEAS International Conference on Applied Mathematics*, (2007), pp. 157–161
21. M. Demiralp, Illustrative implementations to show how logarithm based high dimensional model representation works for various function structures. WSEAS Trans. Comput. **5**, 1339–1344 (2006)
22. M. Demiralp, Plain and logarithmic high dimensional model representation and the effect on their types on univariance level. WSEAS Trans. Math. **5**, 582–588 (2006)
23. R.L. Burden, J.D. Faires, *Numerical Analysis*. Thomson Brooks/Cole. p. 250 (2005)
24. M. Demiralp, No fluctuation approximation in any desired precision for univariate function matrix representations. J. Math. Chem. **47**, 99–110 (2009)
25. N. Altay, M. Demiralp, Numerical solution of ordinary differential equations by Fluctuationlessness theorem. J. Math. Chem. **47**, 1323–1343
26. J.C. Butcher, Coefficients for the study of Runge–Kutta integration processes. J. Austral. Math. Soc. **3**, 185–201 (1963)
27. J.C. Butcher, *Numerical Methods for Ordinary Differential Equations*, 2nd edn. (John Wiley, New York, 2008)

28. J.C. Butcher, Trees and numerical methods for ordinary differential equations. Numer. Algorithms **53**, 153–170 (2009)
29. E. Hairer, G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems* (Springer Verlag, Berlin, 1993)
30. E. Hairer, G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems* (Springer Verlag, Berlin, 1996)
31. C. Gozukirmizi, M. Demiralp, The application of the fluctuation expansion with extended basis set to numerical integration . WSEAS Trans. Math. **8**, 205–212 (2009)
32. M. Demiralp, A new fluctuation expansion based method for the univariate numerical integration under Gaussian weights. in *Proceedings of the 8th WSEAS International Conference on Applied Mathematics*, (2005), pp. 68–73